

Phase Structure of Gauge Theories on an Interval

Yukihiro Fujimoto^{(a) 1}, Tomoaki Nagasawa^{(b) 2}, Satoshi Ohya^{(c) 3}
and Makoto Sakamoto^{(a) 4}

^(a) *Department of Physics, Kobe University, 1-1 Rokkodai, Nada, Kobe 657-8501, Japan*

^(b) *Tomakomai National College of Technology, 443 Nishikioka, Tomakomai 059-1275, Japan*

^(c) *Harish-Chandra Research Institute, Chhatnag Road, Jhusi, Allahabad 211 019, India*

Abstract

We discuss gauge symmetry breaking in a general framework of gauge theories on an interval. We first derive a possible set of boundary conditions for a scalar field, which are compatible with several consistency requirements. It is shown that with these boundary conditions the scalar field can acquire a nontrivial vacuum expectation value even if the scalar mass square is positive. Any nonvanishing vacuum expectation value cannot be a constant but, in general, depends on the extra dimensional coordinate of the interval. The phase diagram of broken/unbroken gauge symmetry possesses a rich structure in the parameter space of the length of the interval, the scalar mass and the boundary conditions. We also discuss 4d chiral fermions and fermion mass hierarchies in our gauge symmetry breaking scenario.

¹E-mail: 093s121s@stu.kobe-u.ac.jp

²E-mail: nagasawa@gt.tomakomai-ct.ac.jp

³E-mail: ohya@mri.ernet.in

⁴E-mail: dragon@kobe-u.ac.jp

1 Introduction

Despite a great success of the standard model (SM), the Higgs sector still remains a mystery. A full understanding of the role of the Higgs scalar is a key ingredient in constructing realistic models beyond the SM. The Higgs scalar of the SM plays two roles: A nonvanishing vacuum expectation value (VEV) of the Higgs field breaks the electroweak gauge symmetry and produces the fermion masses. To break the gauge symmetry, a negative mass square of the Higgs boson is required. Since fermion mass terms are kinematically forbidden in the SM, all the fermions (probably except for neutrinos) acquire their masses through the Yukawa couplings to the Higgs scalar. Therefore, the fermion mass hierarchy originates in the Yukawa coupling one. The purpose of this paper is to discuss gauge symmetry breaking, the chiral structure of fermions and Yukawa coupling hierarchies from a viewpoint of gauge theories on an interval.

An attractive mechanism for generating the Yukawa coupling hierarchy has been proposed by Arkani-Hamed and Schmaltz (AS) [1]. It is naturally realized by localizing the SM fermions at different points in one (more) extra dimension(s), which can be done by coupling five-dimensional fermions to a scalar field with a kink background configuration [2, 3]. The AS mechanism has been extended by further requiring that the Higgs scalar VEV is localized in one extra dimension [4, 5]. The resulting fermion masses are determined by exponentially suppressed overlaps of their wavefunctions and become automatically hierarchical. In order for the above scenario to work, it is important to realize a mechanism to generate an extra dimensional coordinate dependent VEV of a scalar field as a ground state configuration. A mechanism to break translational invariance by a scalar VEV has been proposed in Refs. [6]-[8]. The idea is to impose nontrivial boundary conditions (BC's) incompatible with a nonvanishing constant configuration of a scalar field. The mechanism has been used to break supersymmetry [9, 10] and extended to higher extra dimensions [11, 12]. A similar mechanism to produce a nontrivial profile of a scalar field has been found in orbifold models that orbifold BC's can be chosen to forbid a nonvanishing constant scalar configuration [13]. Another mechanism is to prepare a brane localized potential in addition to a bulk potential in such a way that a minimum configuration of the bulk potential conflicts with that of the brane one [5]. This will force a scalar VEV to depend on the extra dimensional coordinate. The Higgs scalar can also develop a nontrivial profile along the extra dimension by introducing a coupling to another scalar of the localizer [4, 5].

In this paper, we try to find an answer to naturally explain the physics of the Higgs sector in a framework of gauge theories on an interval. We assume that all fields live on the bulk [14] with no brane/boundary localized term, and that any model in our setting

is specified by a bulk Lagrangian and BC's for fields. Since BC's at the boundaries of the interval are crucially important in our framework, we first derive a general class of possible BC's for a scalar field on an interval, which are compatible with several consistency requirements. Those BC's of a scalar are wider than the commonly used orbifold BC's [15]-[17] and characterized by two parameters. In this general setting of the BC's, we find that the scalar field can acquire a nonvanishing vacuum expectation value even if the scalar mass square is positive with no boundary localized terms. Furthermore, the VEV turns out to inevitably depend on the extra dimensional coordinate of the interval. As an illustrative example, we consider a scalar QED on an interval and show that the phase diagram of the broken/unbroken gauge symmetry has a rich structure, which complicatedly depends on the length of the interval, the scalar mass square and the parameters specifying the BC's.

We continue to derive consistent BC's for a fermion on an interval and find that only the four types of BC's are allowed. The type $(++)$ or $(--)$ BC leads to a 4d massless chiral fermion even if the fermion has a bulk mass, while the type $(+-)$ or $(-+)$ BC produces no massless chiral fermions. Thus, only fermions which obey the type $(\pm\pm)$ BC's survive at low energies as 4d massless chiral fermions. All other Kaluza-Klein modes will be decoupled from the low energy spectrum with masses of the order of L^{-1} , which is the inverse of the length of the interval. An important observation is that the profile of a chiral zero mode is exponentially localized at one of the boundaries of the interval. Since chiral fermions could acquire their masses through the Yukawa couplings to the Higgs scalar, it will not be surprising that fermions get hierarchically different masses through the Yukawa interactions because of exponentially localized profiles of chiral zero modes as well as the Higgs VEV with a nontrivial extra dimensional coordinate dependence. Thus, our setting of gauge theories on an interval may be regarded as an explicit realization of the scenario given in Refs. [4, 5]. The above observations strongly suggest that a mystery of the Higgs sector in the SM can be naturally solved in a framework of gauge theories on an interval. Our considerations will mostly be restricted to abelian gauge theories in this paper. However, our mechanism to break gauge symmetries can work for nonabelian gauge theories, as well.

This paper is organized as follows. In the next section, we determine a general consistent set of BC's for a scalar field on an interval. In Section 3, we investigate a scalar QED on an interval, as a demonstration of our symmetry breaking mechanism, and show that the scalar can acquire a nontrivial VEV even if the mass square of the scalar is positive. Furthermore, we clarify a rich phase structure of the model. In Section 4, we consider a fermion on an interval and show a possible class of BC's, in which some of the BC's lead to 4d massless chiral fermions. The Section 5 is devoted to conclusions.

2 Consistent BC's of a Scalar Field

In this section, we investigate a complex scalar on an interval and clarify a class of general consistent BC's for the scalar field. Since the BC's for scalars play a crucial role in our mechanism to break gauge symmetries, we shall discuss the consistency of the allowed BC's from various different points of view. To this end, let us consider a complex scalar field $\Phi(x, y)$ on an interval with an action

$$S = \int d^4x \int_0^L dy \left\{ \Phi^* \partial^\mu \partial_\mu \Phi + \Phi^* \partial_y^2 \Phi - V(|\Phi|^2) \right\}, \quad (1)$$

where $x^\mu (\mu = 0, 1, 2, 3)$ denotes the coordinate of the four-dimensional Minkowski space-time and y is the coordinate of the extra dimension with $0 \leq y \leq L$. Here, the 5d metric is chosen as $\eta_{KN} = \text{diag}(-, +, +, +, +)$.

In one-dimensional quantum mechanics, the most general BC's of a wavefunction are known to be characterized by $U(2)$ parameters at a boundary or point singularity. [18]-[20] If the probability current is required to vanish at a boundary, the $U(2)$ parameters reduce to a subfamily of $U(2)$ at each boundary. Since an interval has two boundaries, at which the probability current has to vanish in order to preserve the probability conservation, the allowed boundary conditions on an interval are found to be given by the Robin boundary condition⁵

$$\begin{aligned} \Phi(0) + L_+ \partial_y \Phi(0) &= 0, \\ \Phi(L) - L_- \partial_y \Phi(L) &= 0, \end{aligned} \quad (2)$$

where L_\pm are arbitrary real constants of mass dimension -1 .

The boundary conditions (2) can also be obtained from the hermiticity requirement of the action, which is necessary to ensure the unitarity of the theory. The condition $S^\dagger = S$ immediately leads to

$$j(y) \equiv -i \left(\Phi^*(y) \partial_y \Phi(y) - (\partial_y \Phi^*(y)) \Phi(y) \right) = 0 \quad \text{at } y = 0, L, \quad (3)$$

where we have assumed that the field and its derivatives become zero at $|x^\mu| \rightarrow \infty$, as usual. The equations (3) can be solved by rewriting it as [21]

$$|\Phi - iL_0 \partial_y \Phi|^2 = |\Phi + iL_0 \partial_y \Phi|^2 \quad \text{at } y = 0, L, \quad (4)$$

where L_0 is an arbitrary nonzero real constant of mass dimension -1 . The above equations imply that $\Phi - iL_0 \partial_y \Phi$ should be proportional to $\Phi + iL_0 \partial_y \Phi$ at $y = 0, L$ and the

⁵ In order to concentrate on the extra dimensional coordinate y , we will omit the x^μ dependence unless otherwise stated.

proportional constants are given by phase factors. Thus, we find that $\Phi - iL_0\partial_y\Phi = e^{i\theta_{0,L}}(\Phi + iL_0\partial_y\Phi)$ at $y = 0, L$ and obtain Eq.(2) with the identification $L_+ = L_0 \cot \frac{\theta_0}{2}$ and $L_- = -L_0 \cot \frac{\theta_L}{2}$.

Another way to derive the BC's (2) is to impose the conservation of a $U(1)$ charge. The action (1) is invariant under the global $U(1)$ transformation: $\Phi \rightarrow e^{i\alpha}\Phi$, and the $U(1)$ current $j_N \equiv -i(\Phi^*\partial_N\Phi - (\partial_N\Phi^*)\Phi)$ ($N = 0, 1, 2, 3, y$) will be conserved, i.e. $\partial^N j_N = 0$. However, this does not necessarily assure the conservation of the $U(1)$ charge

$$Q \equiv \int d^3\mathbf{x} \int_0^L dy j^0(x, y), \quad (5)$$

because

$$\frac{dQ}{dt} = - \int d^3\mathbf{x} \int_0^L dy \partial_y j_y = - \int d^3\mathbf{x} (j_y(x, L) - j_y(x, 0)). \quad (6)$$

Thus, the $U(1)$ charge conservation can be achieved only when the extra dimensional component of the current j_y vanishes at the boundaries $y = 0$ and L . This is identical to the conditions (3), so that the previous argument shows that the BC's (2) guarantee the conservation of the $U(1)$ charge. We should emphasize that the conservation of the global $U(1)$ charge is very important, otherwise the model cannot be extended to a local gauge invariant theory on an interval.

As was proposed in Ref. [22], consistent BC's will be obtained from the action principle. For any infinitesimal variations of Φ , the requirement $\delta S = 0$ leads to the bulk field equation for Φ (or Φ^*), together with the boundary equations

$$\Phi^*\partial_y\delta\Phi - (\partial_y\Phi^*)\delta\Phi = 0 \quad \text{at } y = 0, L. \quad (7)$$

Since Φ^* and $\delta\Phi$ can be independent each other, the above conditions seem to be more restrictive than Eq.(3). This is not, however, the case. The equation (7) is found to lead to the same BC's (2). Indeed, it is easy to see that Eq.(7) is satisfied if both Φ and $\delta\Phi$ obey the BC's (2).

Before closing this section, we should make a comment on the form of the action (1). We could start with the action

$$S' = \int d^4x \int_0^L dy \left\{ -\partial_\mu\Phi^*\partial^\mu\Phi - \partial_y\Phi^*\partial_y\Phi - V(|\Phi|^2) \right\}, \quad (8)$$

instead of Eq.(1). Then, the action principle will lead to the BC's

$$(\partial_y\Phi^*)\delta\Phi = 0 \quad \text{at } y = 0, L, \quad (9)$$

which would require the BC's⁶

$$\partial_y \Phi = 0 \quad \text{or} \quad \Phi = 0 \quad \text{at} \quad y = 0, L. \quad (10)$$

These are special cases (i.e. $L_\pm = \infty$ or $L_\pm = 0$) of the BC's (2). In order to obtain the BC's (2), we may add the following boundary terms

$$\int d^4x \int_0^L dy \left\{ - \left(-\frac{1}{L_+} \delta(y) - \frac{1}{L_-} \delta(y-L) \right) |\Phi(y)|^2 \right\}, \quad (11)$$

to the above action (8). Then, the action principle is found to reproduce the BC's (2). In this point of view, physical meanings of the parameters L_\pm become clear. For $L_\pm > 0$ ($L_\pm < 0$), the terms in Eq.(11) correspond to attractive (repulsive) δ -function potentials at the boundaries with the couplings $1/L_\pm$. This interpretation will be helpful to understand the phase structure of gauge symmetry breaking, as we will discuss later.

3 Phase Structure of Scalar QED on an Interval

In this section, we investigate a scalar QED on an interval, as an illustrative example of our symmetry breaking mechanism, and show that our mechanism possesses notable properties different from the standard Higgs mechanism. As was noted in the Introduction, we assume that all the fields live in the bulk without any brane/boundary localized term, and that the scalar field is allowed to obey a class of the general BC's (2).

3.1 Scalar QED Action and BC's

The action we consider is

$$S = \int d^4x \int_0^L dy \left\{ -\frac{1}{4} F_{KN} F^{KN} + \Phi^* D_\mu D^\mu \Phi + \Phi^* D_y^2 \Phi - M^2 \Phi^* \Phi - \frac{\lambda}{4} (\Phi^* \Phi)^2 \right\}, \quad (12)$$

where $F_{KN} = \partial_K A_N - \partial_N A_K$ and

$$D_N \Phi = (\partial_N - ie A_N) \Phi, \quad N = 0, 1, 2, 3, y. \quad (13)$$

As discussed in the previous section, we take the following BC's for the scalar:

$$\Phi(0) + L_+ \partial_y \Phi(0) = \Phi(L) - L_- \partial_y \Phi(L) = 0. \quad (14)$$

⁶ The equation (9) would allow the BC $\Phi = \text{const}$ at $y = 0, L$ [22]. Any nonvanishing constant value of Φ at the boundaries, however, turns out to be inconsistent with gauge invariance when the scalar field is coupled to a gauge field. This is a signal of the violation of unitarity [23]. Thus, we do not consider this possibility in this paper, although there is an argument that such a boundary condition probably gives a consistent unitary theory [24].

For the gauge fields A_M , we choose the BC's to be of the form

$$\partial_y A_\mu(0) = \partial_y A_\mu(L) = 0, \quad (15)$$

$$A_y(0) = A_y(L) = 0. \quad (16)$$

Since we are interested in gauge symmetry breaking through a nontrivial VEV of Φ , the BC's for the gauge fields have to be chosen not to break the 4d gauge symmetry explicitly. In fact, the BC's (15) allow a massless zero mode of the 4d gauge field, as they should be.

It is important to verify that the BC's (15) and (16) are consistent with the 5d gauge transformations:

$$\Phi(x, y) \rightarrow \Phi'(x, y) = e^{ie\Lambda(x, y)} \Phi(x, y), \quad (17)$$

$$A_\mu(x, y) \rightarrow A'_\mu(x, y) = A_\mu(x, y) + \partial_\mu \Lambda(x, y), \quad (18)$$

$$A_y(x, y) \rightarrow A'_y(x, y) = A_y(x, y) + \partial_y \Lambda(x, y). \quad (19)$$

It follows from the transformation (18) that the gauge parameter $\Lambda(x, y)$ should obey the same BC's as $A_\mu(x, y)$, i.e.

$$\partial_y \Lambda(x, 0) = \partial_y \Lambda(x, L) = 0, \quad (20)$$

which is consistent with the transformation (19) and the BC's (16) for A_y . Note that the compatibility between the BC's (15) and (16) can also be shown from a viewpoint of quantum mechanical supersymmetry. [21, 25]–[27]

The consistency of the BC's (14) for Φ with the gauge transformation (17) requires that Φ' given in Eq.(17) should obey the same BC's as the original field Φ . This can be verified as follows:

$$\Phi'(y) \pm L_\pm \partial_y \Phi'(y) = e^{ie\Lambda(y)} \left\{ \left(\Phi(y) \pm L_\pm \partial_y \Phi(y) \right) \pm ie(\partial_y \Lambda(y)) L_\pm \Phi(y) \right\}. \quad (21)$$

It is now easy to see that $\Phi(y)$ and $\Phi'(y)$ satisfy the same BC's (14) with the conditions (20).

It should be further noticed that the BC's (14), (15) and (16) satisfy all the requirements discussed in the previous section. The action (12) is hermitian and the $U(1)$ charge is conserved (if the gauge symmetry is unbroken). The boundary equations derived from the action principle are satisfied for the BC's chosen here.

3.2 Phase Structure

In this subsection, we would like to determine whether or not the scalar field Φ can acquire a nonzero VEV. In order to find the vacuum configuration, one might try to minimize the

potential $V = M^2|\Phi|^2 + \frac{\lambda}{4}|\Phi|^4$. This is, however, wrong in the present model. It turns out that the vacuum configuration is given by solving the minimization problem of the functional

$$\mathcal{E}[\Phi] \equiv \int_0^L dy \left\{ \Phi^* (-\partial_y^2) \Phi + M^2 |\Phi|^2 + \frac{\lambda}{4} |\Phi|^4 \right\}. \quad (22)$$

The important point is to incorporate the kinetic term along the extra dimension into the effective 4d potential $\mathcal{E}[\Phi]$, because the minimum configuration of Φ can have the y dependence, as we will see below.

In the following, we will ignore the x^μ dependence in Φ , which is irrelevant in our analysis since we assume the translational invariance of the 4d Minkowski spacetime. Since we are interested in the gauge symmetry breaking, we would like to know whether or not the vacuum configuration $\langle \Phi(y) \rangle$ is nonvanishing. To this end, it is convenient to introduce the eigenfunctions $f_n(y)$ of the eigenvalue equation

$$-\partial_y^2 f_n(y) = E_n f_n(y), \quad n = 0, 1, 2, \dots, \quad (23)$$

with the BC's

$$f_n(0) + L_+ \partial_y f_n(0) = f_n(L) - L_- \partial_y f_n(L) = 0. \quad (24)$$

In terms of the orthonormal eigenfunctions f_n , the field Φ can be expanded as

$$\Phi(y) = \sum_{n=0}^{\infty} \phi_n f_n(y). \quad (25)$$

Inserting it into $\mathcal{E}[\Phi]$ leads to

$$\mathcal{E}[\Phi] = \sum_{n=0}^{\infty} m_n^2 |\phi_n|^2 + (\text{quartic terms in } \phi_n), \quad (26)$$

where

$$m_n^2 \equiv M^2 + E_n, \quad n = 0, 1, 2, \dots. \quad (27)$$

Note that the quartic terms are non-negative for any configurations of ϕ_n because they come from the term $\int dy \frac{\lambda}{4} |\Phi|^4 \geq 0$. It immediately follows that the vacuum configuration is given by $\langle \Phi \rangle = 0$ (or $\langle \phi_n \rangle = 0$ for any n) if $m_n^2 \geq 0$ for any n . Actually, we are interested in the inverse of the above statement that Φ (or ϕ_0) acquires a nontrivial VEV if $m_0^2 < 0$ for the lowest eigenvalue E_0 . This implies that the gauge symmetry breaking can occur even for $M^2 > 0$ if a bound state exists in the spectrum. On the other hand, the gauge symmetry can still be unbroken even for $M^2 < 0$ if there is no bound state with $E_0 > 0$.

Therefore, in order to determine whether or not the gauge symmetry breaking occurs, we need to solve the bound state problem of Eq.(23) with the BC's (24) and find the lowest energy eigenvalue E_0 .

We can assume any bound state solution $f_{E<0}(y)$ and any positive energy solution $f_{E>0}(y)$, without loss of generality, to be of the form⁷

$$f_{E<0}(y) = a e^{\kappa(y-L/2)} + b e^{-\kappa(y-L/2)} \quad \text{with } E = -\kappa^2 < 0, \quad (28)$$

$$f_{E>0}(y) = A e^{ik(y-L/2)} + A^* e^{-ik(y-L/2)} \quad \text{with } E = k^2 > 0, \quad (29)$$

where a, b, κ and k are real numbers with $\kappa, k > 0$ and A is a complex one. Inserting the above expressions into the BC's (24) and requiring nontrivial solutions to exist, we find the equations to determine the energy spectrum, i.e.

$$\tanh(\kappa L) = \frac{\kappa(L_+ + L_-)}{1 + \kappa^2 L_+ L_-}, \quad (30)$$

$$\tan(kL) = \frac{k(L_+ + L_-)}{1 - k^2 L_+ L_-}. \quad (31)$$

As was mentioned above, the criterion of the gauge symmetry breaking is

$$m_0^2 = M^2 + E_0 < 0 \quad (32)$$

for the lowest eigenvalue E_0 . Noting that the parameters L_{\pm} appear only in the symmetric combinations $L_+ L_-$ and $L_+ + L_-$ in the transcendental equations (30) and (31), we find four distinct patterns of the spectrum, according to the signs of $L_+ L_-$ and $L_+ + L_-$. For the following discussions, it is convenient to introduce the maximum and minimum values of the set $\{L_+, L_-\}$

$$L_{\max} \equiv \max\{L_+, L_-\}, \quad L_{\min} \equiv \min\{L_+, L_-\}. \quad (33)$$

(a) $L_+ L_- > 0$ and $L_+ + L_- > 0$

Let us first consider the case of $L_{\pm} > 0$, which may be interpreted as the presence of two attractive δ -function potentials at the boundaries. In this case, there exist two bound states for $L > L_+ + L_-$ and a single one for $L \leq L_+ + L_-$. The lowest eigenvalue is found to satisfy $E_0 < -1/(L_{\min})^2$ (see Fig.1(a)). It follows that the gauge symmetry is spontaneously broken for $M^2 < 1/(L_{\min})^2$, because $m_0^2 = M^2 + E_0 < 0$. For $M^2 > 1/(L_{\min})^2$, there exists a critical length L_c defined by

$$L_c = \frac{1}{|M|} \operatorname{arctanh} \left(\frac{|M|(L_+ + L_-)}{1 + M^2 L_+ L_-} \right) \quad \text{for } M^2 > \frac{1}{(L_{\min})^2}, \quad (34)$$

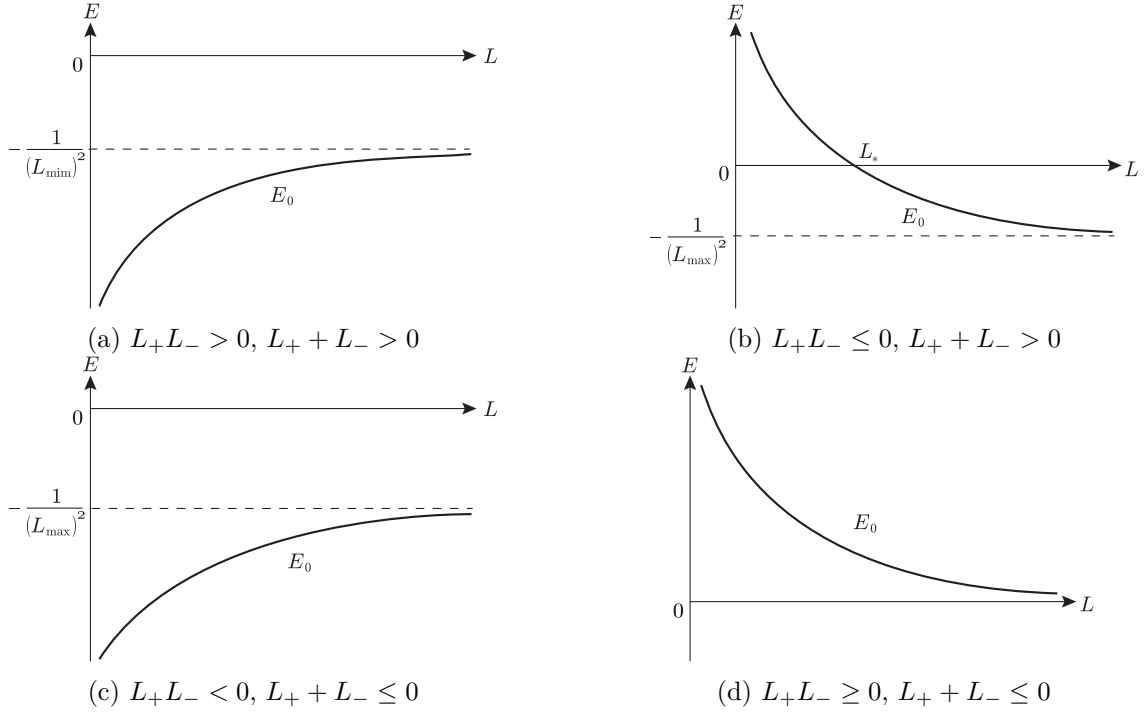


Figure 1: The lowest energy spectrum E_0 . L_* is given by $L_* = L_+ + L_-$.

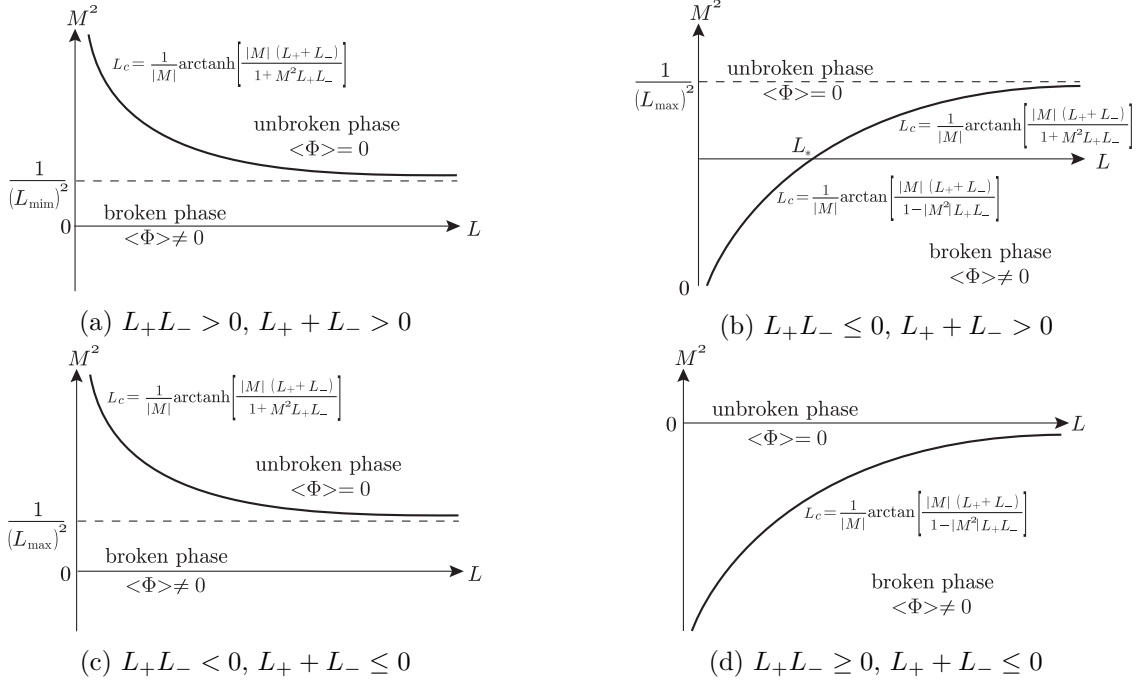


Figure 2: Phase diagrams on an interval.

and the gauge symmetry is broken (unbroken) for $L < L_c$ ($L \geq L_c$). The phase diagram is schematically depicted in Fig.2(a).

(b) $L_+L_- \leq 0$ and $L_+ + L_- > 0$

Let us next consider the case of $L_{\max} > 0$ and $L_{\min} \leq 0$ with $L_{\max} > |L_{\min}|$, which may be interpreted as the presence of a relatively weak attractive δ -function potential and a relatively strong repulsive δ -function potential at the boundaries. In this case, there is a bound state for $L > L_* \equiv L_+ + L_-$ with $0 > E_0 > -1/(L_{\max})^2$. For $L < L_*$, there is no bound state and the lowest energy E_0 is positive (see Fig.1(b)). Thus, for $M^2 \geq 1/(L_{\max})^2$, m_0^2 is always non-negative and hence the gauge symmetry is unbroken. For $M^2 < 1/(L_{\max})^2$, the gauge symmetry is broken (unbroken) for $L > L_c$ ($L \leq L_c$), where the critical length L_c is defined by

$$L_c = \begin{cases} \frac{1}{|M|} \operatorname{arctanh} \left(\frac{|M|(L_+ + L_-)}{1 + M^2 L_+ L_-} \right) & \text{for } 0 < M^2 < \frac{1}{(L_{\max})^2}, \\ \frac{1}{|M|} \operatorname{arctan} \left(\frac{|M|(L_+ + L_-)}{1 - |M|^2 L_+ L_-} \right) & \text{for } M^2 < 0, \end{cases} \quad (35)$$

Here, the values of $\operatorname{arctan}(x)$ should be chosen in the range of $0 < \operatorname{arctan}(x) < \pi$. The resulting phase diagram is schematically depicted in Fig.2(b).

(c) $L_+L_- < 0$ and $L_+ + L_- \leq 0$

Let us next consider the case of $L_{\max} > 0$ and $L_{\min} < 0$ with $L_{\max} \leq |L_{\min}|$, which may be interpreted as the presence of a relatively strong attractive δ -function and a relatively weak repulsive δ -function potential at the boundaries. In this case, there is a single bound state in the spectrum with $E_0 < -1/(L_{\max})^2$ (see Fig.1(c)).⁸ Thus, for $M^2 < 1/(L_{\max})^2$, m_0^2 is always negative, and hence the gauge symmetry is broken. For $M^2 > 1/(L_{\max})^2$, the gauge symmetry is broken (unbroken) for $L < L_c$ ($L \geq L_c$), where the critical length is defined by

$$L_c = \frac{1}{|M|} \operatorname{arctanh} \left(\frac{|M|(L_+ + L_-)}{1 + M^2 L_+ L_-} \right) \quad \text{for } M^2 > \frac{1}{(L_{\max})^2}. \quad (36)$$

The resulting phase diagram is schematically depicted in Fig.2(c).⁹

(d) $L_+L_- \geq 0$ and $L_+ + L_- \leq 0$

⁷ Zero energy solutions are given by $f_{E=0}(y) = a + by$.

⁸ For the case of $L_+ + L_- = 0$, E_0 is given by $E_0 = -1/(L_{\max})^2$, which may be obtained by taking the limit of $L_+ + L_- \rightarrow -0$.

⁹ For $L_+ + L_- = 0$, the critical line is given by $M^2 = 1/(L_{\max})^2$.

Let us finally consider the case of $L_{\pm} \leq 0$, which may be interpreted as the presence of two repulsive δ -function potentials at the boundaries. In this case, there is no bound state and $E_0 > 0$ (see Fig.1(d)). Thus, for $M^2 > 0$, m_0^2 is always positive and hence the gauge symmetry is unbroken. An interesting observation is that even if M^2 is negative, the gauge symmetry is unbroken for $L > L_c$, where the critical length is defined by

$$L_c = \frac{1}{|M|} \arctan\left(\frac{|M|(L_+ + L_-)}{1 - |M|^2 L_+ L_-}\right) \quad \text{for } M^2 < 0. \quad (37)$$

Here, the values of $\arctan(x)$ should be chosen in the range of $0 < \arctan(x) < \pi$. The resulting phase diagram is schematically depicted in Fig.2(d).

In the above analysis, we have succeeded to determine the broken/unbroken phases of the gauge symmetry in the parameter space of the theory. To derive them, we did not need the exact value of $\langle \Phi(y) \rangle$ but it was sufficient to know whether or not $\langle \Phi(y) \rangle$ is nonvanishing. However, the exact value will be required to obtain the gauge boson mass in the broken phase and masses of the Kaluza-Klein modes of the scalar field. According to similar analyses given in Refs. [28, 29, 6] with the different BC's, we can show that the exact VEV of $\langle \Phi(y) \rangle$ is given in terms of Jacobi elliptic functions and that the phase diagrams are precisely reproduced. The Kaluza-Klein mass spectrum of the scalar field turns out to be governed by Lamé-type equations. The details will be reported in a separate paper [30].

4 Fermion BC's and 4d Chiral Zero Mode

In the previous section, we have succeeded to reveal a rich phase structure of the scalar QED on the interval. We would like to extend our analysis to gauge theories coupled to fermions. To this end, we add the fermionic terms

$$S_F = \int d^4x \int_0^L dy \bar{\Psi}(iD_\mu \gamma^\mu + iD_y \gamma^y + M_F)\Psi \quad (38)$$

to the action (12). Here, $\Psi(x, y)$ is a 4-component Dirac spinor and

$$D_N \Psi = (\partial_N - ie_F A_N) \Psi. \quad (39)$$

Note that the bulk fermion mass M_F should be included in Eq.(38) because there is no Weyl fermion in 5-dimensions. The extra component γ^y of the gamma matrices can be chosen as

$$\gamma^y = -i\gamma^5 = \gamma^0 \gamma^1 \gamma^2 \gamma^3. \quad (40)$$

To obtain consistent BC's for Ψ , we require the action principle¹⁰ and find that $\bar{\Psi}\gamma^y\delta\Psi = 0$ at $y = 0, L$. The condition may be decomposed into $\bar{\Psi}_+\delta\Psi_- = \bar{\Psi}_-\delta\Psi_+ = 0$ at $y = 0, L$, where Ψ_\pm are chiral spinors defined by $\gamma^5\Psi_\pm = \pm\Psi_\pm$. It follows that the consistent BC's for a fermion are found to be

$$\Psi_+ = 0 \quad \text{or} \quad \Psi_- = 0 \quad \text{at } y = 0, L. \quad (41)$$

The action principle also leads to the bulk equation, i.e. the 5d Dirac equation $(iD_\mu\gamma^\mu + D_y\gamma^5 + M_F)\Psi = 0$. In terms of Ψ_\pm , the Dirac equation is decomposed as

$$iD_\mu\gamma^\mu\Psi_+ + (-D_y + M_F)\Psi_- = 0, \quad (42)$$

$$iD_\mu\gamma^\mu\Psi_- + (D_y + M_F)\Psi_+ = 0. \quad (43)$$

The above equations imply that $\Psi_+ = 0$ ($\Psi_- = 0$) at $y = 0$ or L automatically gives the BC for Ψ_- (Ψ_+) as $(-D_y + M_F)\Psi_- = 0$ ($(D_y + M_F)\Psi_+ = 0$) at $y = 0$ or L . We thus conclude that the fermion should obey one of the following four BC's:

$$\begin{aligned} \text{type}(++) : (D_y + M_F)\Psi_+ = \Psi_- = 0 & \quad \text{at } y = 0 \text{ and } L, \\ \text{type}(--) : \Psi_+ = (-D_y + M_F)\Psi_- = 0 & \quad \text{at } y = 0 \text{ and } L, \\ \text{type}(+-) : (D_y + M_F)\Psi_+ = \Psi_- = 0 & \quad \text{at } y = 0, \\ \Psi_+ = (-D_y + M_F)\Psi_- = 0 & \quad \text{at } y = L, \\ \text{type}(-+) : \Psi_+ = (-D_y + M_F)\Psi_- = 0 & \quad \text{at } y = 0, \\ (D_y + M_F)\Psi_+ = \Psi_- = 0 & \quad \text{at } y = L. \end{aligned} \quad (44)$$

This result is very important in constructing phenomenological models on an interval because a chiral 4d fermion ψ_+ (ψ_-) appears in the 4d spectrum if a 5d fermion obeys the type(++) (type(--)) BC, while all the fermions with type($\pm\mp$) BC's will be decoupled from the low energy spectrum. It is important to note that chiral 4d fermions are exponentially localized at boundaries. With the type(++) BC, every 4d chiral zero mode is localized at $y = 0$ for $M_F > 0$ ($y = L$ for $M_F < 0$) according to the profile $\sim e^{-M_F y}$. On the other hand, with the type(--) BC, every 4d chiral zero mode is localized at $y = L$ for $M_F > 0$ ($y = 0$ for $M_F < 0$) according to the profile $\sim e^{M_F y}$. It should be emphasized that the bulk mass M_F has nothing to do with the presence or absence of a chiral zero mode but affects its profile, and also that the analysis with the introduction of Yukawa interactions will be performed in a similar way.

To get a phenomenological model, we need to extend our analysis to nonabelian gauge theories. Our mechanism to break gauge symmetries still works for them. A simple

¹⁰ The other requirements, such as the hermiticity of the action and the fermion number conservation, will lead to the same conclusion.

extension of the SM, as a starting point to construct a realistic model beyond the SM, may be given as follows. The 4d gauge fields of the SM are replaced by the 5d gauge fields with the BC's similar to Eqs.(15) and (16),¹¹ which are consistent with 4d gauge symmetries of the SM.¹² The 4d chiral fermions of the SM have to be replaced by 5d Dirac fermions with their bulk masses. Assuming that the 5d fermions have the same quantum numbers as the SM fermions, we impose the type(++) BC for the $SU(2)$ singlet fermions and the type(--) BC for the $SU(2)$ doublet ones. Then, we may have desired 4d chiral fermions of the SM at low energies irrespective of the bulk fermion masses. A key ingredient of our model is the choice of nontrivial BC's (14) for the Higgs field, which generate a nontrivial y -dependent VEV $\langle\Phi(y)\rangle$ and the electroweak gauge symmetry breaking. Since bulk fermion masses do not provide masses of 4d chiral fermions, as was noted above, they should acquire their masses through Yukawa interactions with localized profiles of chiral zero modes and the Higgs VEV $\langle\Phi(y)\rangle$. Thus, we expect the model to mimic the SM at low energies as a simple realization of the scenario given by [4, 5].¹³ The work along this line will be reported elsewhere [30].

5 Conclusions

We have investigated the nature of gauge symmetry breaking in gauge theories coupled with a scalar field on an interval. We first derived the consistent set of boundary conditions for a scalar field. These scalar BC's are characterized by two real parameters. We have checked that they are compatible with the various consistency requirements; the action principle, the gauge invariance, the hermiticity of the action and the charge conservation. Allowing general BC's for the scalar field, we have observed that the scalar can develop a nonvanishing VEV to break gauge symmetry, like the Higgs field of the SM. The gauge symmetry breaking mechanism is, however, quite different from the usual Higgs one. We do not need a negative mass square term to break gauge symmetry. The scalar field can acquire a nontrivial VEV even if its mass square is positive. Any nonvanishing value of the scalar field cannot be a constant but inevitably depends on the extra dimensional coordinate. The phase diagram is found to depend nontrivially on the length of the interval, the mass and the BC's of the scalar field.

Since the main purpose of this paper is to demonstrate our gauge symmetry breaking

¹¹ We will have a more variety of BC's than those considered in this paper in nonabelian gauge theories with many flavors.

¹² Since there are no massless zero modes of the extra components A_y of the gauge fields with the BC (16), the Hosotani mechanism [31, 32, 33] to break gauge symmetry does not work in this model.

¹³ A challenging attempt may be to introduce many branes or point singularities [34, 35] on an interval, in which several copies of chiral fermions will appear. Then, we can attack the generation problem of the SM together with the fermion mass hierarchy one.

mechanism, we have restricted our considerations to a simple $U(1)$ gauge theory. However, the extension to nonabelian gauge theories is straightforward. We will then have a large variety of consistent BC's to break gauge symmetries. As was discussed in Section 4, 4d chiral fermions naturally arise from bulk fermions on an interval even if their bulk masses are nonvanishing, and they are, in general, localized at one of the boundaries. This is good news to solve the fermion mass hierarchy problem. Chiral fermions will acquire their masses through Yukawa couplings. We may then have a chance to get the hierarchical fermion masses of the SM with the localization property of chiral zero modes together with the extra dimensional coordinate dependence of the VEV $\langle\Phi(y)\rangle$.

In this paper, we have discussed the phase structure of gauge theories on an interval at the tree level. Quantum effects may, however, change our results because they will produce mass corrections to the Higgs scalar, which, in general, depend on the scale of the extra dimension. Such radiative corrections would become important when a compactification scale becomes less than the inverse of a typical mass scale of the theory, and then some of broken symmetries could be restored or conversely some of symmetries could be broken, as shown in Ref. [8]. Thus, our analyses at the tree level will be insufficient and it would be of great importance to investigate quantum corrections in a class of theories we considered.

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